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A Separable Goal Programming Approach to Optimizing Multivariate Sampling Designs for Forest Inventory

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ABSTRACT. Describes the application of a separable goal programming approach to stratified random sampling involving multiple objectives. Other attempts at solving this problem are also reviewed. The method is applied to a forest inventory problem in New Mexico involving six objectives and fourteen strata. Eight sampling allocations are presented to illustrate the sensitivity to alternate preference functions. Intercorrelations among goal criteria limit the effects of alternative preferences on resulting sampling allocations. All sampling allocations are guaranteed to be nondominated—something that goal programming does not (in general) provide. FOREST SCI. 27:147–162.

ADDITIONAL KEY WORDS. Stratified random sampling, multiple criteria.

IN RECENT YEARS, interest in applying formal decisionmaking techniques to forest inventory has grown (Burkhart and others 1978, Hamilton 1979). As the costs of forest inventory have risen, "cost-effective" has become a common phrase of inventory designers (Ware 1974, Avery 1974, O'Regan and Arvanitis 1966).

The dual objective of the inventory specialist is to provide valuable information for management at low cost. The means of accomplishing this task is the selection of suitable sampling and estimation procedures for the particular problem at hand. The designer must exploit his knowledge of the intended use of the information, the statistical characteristics of the population(s), and the technical and financial attributes of alternative sampling plans (Ware 1974).

In order to use optimization methods to solve the sampling allocation problem, forest managers must quantify their information goals in terms of cost and precision of sample estimates. In addition, the appropriate loss function involving these goals must be specified either explicitly as a mathematical function or implicitly by preferences between alternative sampling designs. With this quantitative framework, the inventory specialist can help the manager select the most cost-effective sampling plan.

In this paper a technique is presented for selecting an allocation of inventory resources for a stratified random sampling (SRS) design when faced with multiple goals. The method is applied to data from a U.S. Forest Service inventory in northern New Mexico.[1]

[1] USDA Forest Service. 1975. Operating Plan for the forest resource inventory of selected counties in northern New Mexico. Unpublished report, USDA Forest Service, Intermountain Forest and Range Exp Stn, Ogden, Utah.

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The Stratified Random Sampling Problem

Stratified random sampling (SRS) is a commonly used design for obtaining estimates of population characteristics because: (a) separate estimates of the means and variances can be made for each of the forest strata, and (b) for a given sampling intensity, stratification often yields more precise estimates of the forest characteristics than does a simple random sample of the same size. This will be achieved if the sampling variance for a characteristic is significantly lower within the established strata than within the population as a whole. The main disadvantage of stratification is that stratum sizes must be known or fairly accurately estimated if the gain in precision is to be realized (Husch and others 1972).

Stratification is most efficient when strata definitions are based on the characteristic which is to be estimated by the sampling procedure. In a multivariate inventory, the designer would ideally form a different set of strata for each characteristic to be estimated. In practice, cost considerations often limit the basis for stratification to information available from aerial photographs. Since some population characteristics cannot be estimated very well from photos (i.e., growth rates) optimal stratification with respect to these characteristics is infeasible. As a compromise, stand types are often used to delineate strata because they form logical management units.

For simplification it will be assumed that all characteristics are measured on each sampling unit selected from the population. This is reasonable if the marginal cost of measuring additional characteristics is low compared to the cost of plot establishment. Extension to the case of differing sampling intensities is straightforward. However, this increases the size of the allocation problem as a function of the number of population characteristics being estimated.

For SRS in which stratum sizes are known an unbiased estimate of the population mean for a characteristic is

$$\bar{y} = \sum_{h=1}^{L} N_h \bar{y}_h / N \tag{1}$$

with variance (Cochran 1963),

$$V(\bar{y}) = \sum_{h=1}^{L} W_h^2 S_h^2 (1/n_h - 1/N_h) \tag{2}$$

where

$\bar{y}$ = Estimate of population mean
N_h = Total number of sampling units in h^{th} stratum
N = Total number of sampling units in population $\left(\sum_{h=1}^{L} N_h\right)$
$\bar{y}_h$ = Estimate of population mean in h^{th} stratum
$V(\bar{y})$ = Estimate of population variance of $\bar{y}$
W_h = N_h/N = Proportion of population sampling units in h^{th} stratum
S_h^2 = Estimate of variance of y in h^{th} stratum
n_h = Number of sampling units in h^{th} stratum
L = Number of strata.

To develop a cost-effective allocation one must assume a cost function. In this paper only the linear cost function

$$C = C_0 + \sum_{h=1}^{L} c_h n_h \tag{3}$$

will be used. C_0 represents fixed costs and c_h is the cost per sampling unit taken from the hth stratum. Variable travel costs between sampling units are assumed

negligible. While less than realistic, the linear cost function is adopted for simplicity and comparability with previous work (Chatterjee 1968, Hartley 1965, Arvanitis and Afonja 1971). Extension to more complex cost functions is discussed later.

With these assumptions, the SRS problem is one of deciding how many sampling units to take from each stratum. The criteria to be considered in selecting a set of sample sizes are sampling costs (C) and the variances of the sample estimates (set of $V(\bar{y})$'s). For each criterion there is a corresponding objective (goal) such as minimization of each $V(\bar{y})$ and C. The inventory designer seeks to maximize the decisionmaker's satisfaction (minimize his loss function) through a compromise among these conflicting multiple objectives.

Solving Multiple Objective Optimization Problems

The traditional approach to optimization grew out of the neoclassical theory of the firm where a single objective, maximization of profit, was assumed to be a comprehensive measure of utility (Keen 1977). This simplification allowed the development and application of a wide range of mathematical techniques to the problem of finding the (usually unique) optimal solution.

The same methodology has been adapted to problems involving multiple objectives by weighting or otherwise defining a single measure of utility. However, if there are multiple incommensurable objectives, these methods are of little value. Recently this has been recognized as a more realistic view of many management problems. Consequently Multiple Criteria Decision-Making (MCDM) is receiving widespread attention as a current area of study in operations research (Roy 1971, Cochrane and Zeleny 1973, Cohon and Marks 1975, Starr and Zeleny 1977).

In trying to extend the concept of optimality to multiple objectives a problem is encountered—what is meant by minimizing (or maximizing) a vector of objective functions? Rare indeed is the case where a solution exists where all components are simultaneously at their optima. Usually one is faced with tradeoffs between objectives, the satisfaction of one being decreased or increased at the expense of others. This has led to the concept of nondominated or Pareto optimal solutions (Haimes and others 1975).

Consider a MCDM problem with L decision variables and v objective functions. Without loss of generality assume that all objective functions are to be minimized. Define two solutions to the problem by their vectors of decision variables, $X = (x_1, x_2, \ldots, x_L)$ and $Y = (y_1, y_2, \ldots, y_L)$. Let the corresponding objective function vectors be $F_X = [f_1(X), f_2(X), \ldots f_v(X)]$ and $F_Y = [f_1(Y), f_2(Y), \ldots, f_v(Y)]$ for X and Y, respectively. Solution Y is said to be nondominated (or noninferior, efficient, or Pareto optimal) if the following conditions are true for any other solution vector X in the decision space:

$f_j(Y) \leq f_j(X)$ for each objective function j, $j = 1, 2, \ldots, v$; and

$f_j(Y) < f_j(X)$ for at least one of the v objective functions. Since Y is always as good as or better than X in terms of the objective function vector, it is said to dominate X (Haimes and others 1975).

Regardless of the preference structure associated with the multiple objective functions, it is clear that the chosen solution must belong to the set of nondominated solutions (if the set is not empty). Such a solution is labeled the best compromise solution since it usually involves a compromise among the extremizations of the multiple objective functions (Cohon 1978). This best compromise solution can be selected from the nondominated set only if a particular preference structure is specified.

Many techniques and procedures have been suggested for locating the non-

dominated set and/or best compromise solution. Cohon (1978) and Cohon and Marks (1975) classify all multiobjective techniques as: (a) generating procedures, (b) iterative procedures where the preference structure is elucidated in a stepwise process, and (c) noniterative procedures where the preference structure is prespecified.

The generating procedures require identification of the set of all nondominated solutions. Specific techniques within this class of procedures are methods based on weights (Geoffrion 1968), the multicriteria simplex (Zeleny 1974), the constraint method (Haimes 1973) and the adaptive search method (Beeson and Meisel 1971).

Iterative procedures explore the solution space incorporating the preference structure given at each step. Examples of iterative methods are the step method (Benayoun and others 1971), the interactive technique of Dyer (1972), and the contracting cone method (Steuer 1978).

Noniterative procedures refer to those where the preference structure is prespecified prior to initiating the search for the best compromise solution. Examples of techniques within this class of procedures are utility theory (Kenney and Raiffa 1976), goal programming (Ignizio 1976), and the surrogate worth tradeoff method (Haimes and Hall 1974).

In this paper, a goal programming approach is employed to reach a best compromise solution. Special care is taken to insure that this solution is nondominated—something which goal programming in general does not guarantee (Cohon 1978, Dyer and others 1979).

Solving the Multivariate SRS Problem

Using eqs 2 and 3 and following Cochran (1963) one can derive for the univariate case the minimum cost allocation for fixed precision,

$$n_h = W_h S_h \left(\sum_{h=1}^{L} W_h S_h c_h^{\frac{1}{2}} \right) \Big/ \left[c_h^{\frac{1}{2}} \left(V(\bar{y}) + N^{-1} \sum_{h=1}^{L} W_h S_h^2 \right) \right] \tag{4}$$

and the maximum precision allocation for fixed cost,

$$n_h = (C - C_0) W_h S_h \Big/ \left[c_h^{\frac{1}{2}} \left(\sum_{h=1}^{L} W_h S_h c_h^{\frac{1}{2}} \right) \right]. \tag{5}$$

Early approaches to the multivariate case involved compromises among the optimal allocations (as given by eq 5) for each variate. Cochran (1963) and Ghosh (1958) suggested averaging the optimal allocations with respect to each variable. This implicitly leads to an equal weighting of all variates. Dalenius (1953), Chatterjee (1967), and Hartley (1965) utilized loss function approaches with equal or differential weights assigned to the different variables.

The above approaches reduce the multiple objective problem to one with a single optimality criterion through assumptions about the relative importance of the precision of each estimate. With these one-step optimization solutions the allocation of sampling resources is a direct function of the weights assumed. The weakness of these methods is that no procedure for setting the weights has been developed. Further, such a method cannot be developed unless a specific preference structure or loss function is assumed.

Another extension of the univariate case is to minimize the determinant of the covariance matrix of the variate means as first suggested by Ghosh (1958). Arvanitis and Afonja (1971) expressed this determinant as a function of the n_h's for the bivariate and trivariate cases and used nonlinear programming to find the

optimal allocation. Once again, the precision of each estimate was considered equally important.

Several mathematical programming approaches have been used to solve the multiple criteria SRS problem. Consider the general case with L strata for which v population means are to be estimated.

$$\begin{aligned} &\text{Minimize} && C = C_0 + \sum_{h=1}^{L} c_h n_h \\ &\text{subject to} && V(\bar{y}_j) \leq V_j \quad \text{for} \quad j = 1, 2, \ldots, v \\ & && n_h \leq N_h \quad \text{for} \quad h = 1, 2, \ldots, L. \end{aligned} \tag{6}$$

The first set of constraints specify the upper allowable bounds on the variances of the estimates while the second set prevent the sample size selected (n_h) from exceeding the stratum size (N_h). Recall from eq 2 that $V(\bar{y}_j)$ involves the reciprocals of the n_h's and thus the first set of constraints are nonlinear in the decision variables.

In an attempt to force this nonlinear problem into a linear programming framework, Nordbotten (1956) altered the objective function and rearranged the second set of constraints to obtain the following SRS problem where the reciprocals of the n_h's become the decision variables,

$$\begin{aligned} &\text{Minimize} && C = C_0 + \sum_{h=1}^{L} c_h / n_h \\ &\text{subject to} && V(\bar{y}_j) \leq V_j \quad \text{for} \quad j = 1, 2, \ldots, v \\ & && N_h^{-1} - n_h^{-1} \leq 0 \quad \text{for} \quad h = 1, 2, \ldots, L. \end{aligned} \tag{7}$$

This formulation is of questionable utility because total cost decreases as the number of samples taken in a stratum increases.

Before using the general SRS problem formulated in eq 6, the question of its feasibility must be addressed. Because of the convex objective function (in the reciprocals of the n_h's), an optimum solution exists if it can be shown that a solution exists (Kokan 1963). Kokan and Khan (1967) have proved the existence and uniqueness of such a solution to eq 6. Thus, the feasibility of eq 6 is assured.

The general SRS problem formulated in eq 6 was also solved by Chatterjee (1968) using an undescribed algorithm. Since the V_j's were arbitrary, a linear interpolation of a Taylor series expansion of the objective (cost) function about the V_j's was used to find approximate costs for alternative sets of precision levels. Thus a neighborhood of the decision space was explored (approximately) without solving more than one such problem.

Hartley (1965) suggested that linear programming be used to solve an approximation to eq 6 in which the cost function was replaced with a piecewise linear approximation. This was possible because the objective function was separable permitting solution by separable programming (Charnes and Lemke 1954).

The main drawback with mathematical programming approaches to the SRS problem is that there is no assurance that the optimal allocation will be nondominated. In fact, for certain values of the V_j's the formulated problem may have no feasible solution. Conceptually there is a lack of consistency in the treatment of the evaluative criteria; costs are minimized while the variances of the estimates are constrained by fixed upper bounds.

Fortunately, these methods can be easily adapted to generating the entire nondominated solution set by parametrically varying the objective function weights or constraint levels. Though each such formulation can be solved by convex

programming techniques (Hazard and Promnitz 1974), the cost in computing time can make explorations of the solution space prohibitively expensive. Of course one could simply calculate precision and cost for all permutations of sample sizes but again this can be quite expensive and does not point towards any particular solution as the best compromise.

Only Folks and Antle (1965) have considered the SRS problem from the Pareto optimality point of view. They simplified the problem slightly by assuming that sampling costs were the same for each stratum. Then, for a fixed total sample size of n, the complete set of nondominated allocations for the SRS problem is generated by

$$n_h = \left[N_h \cdot n\left(\sum_{j=1}^{v} \lambda_j S^2_{jh}\right)^{\frac{1}{2}}\right] \Big/ \left[\sum_{h=1}^{L} N_h\left(\sum_{j=1}^{v} \lambda_j S^2_{jh}\right)^{\frac{1}{2}}\right] \tag{8}$$

where the λ_j's are arbitrary weighting constants such that

$$\sum_{j=1}^{v} \lambda_j = 1 \quad \text{and} \quad \lambda_j \geqslant 0 \quad \text{for} \quad j = 1, 2, \ldots, v. \tag{9}$$

Unfortunately, no technique for finding a best compromise solution within the nondominated set was presented.

Clearly, any allocation of the n_h's must be nondominated unless some of the cost coefficients (c_h) or stratum variances (S_h^2) are zero. By eqs 2 and 3, any change in an n_h changes both the precision of every estimator and the total cost unless one of the above degeneracies occurs. Thus, there is usually a tradeoff between precision and cost.

Applying Goal Programming to the SRS Problem

Goal programming is a variation of linear programming which allows the incorporation of multiple objectives. This is accomplished by setting goals or attainment levels for each objective and then minimizing the weighted sum of deviations from these goals. In linear programming only one criterion is used in the objective function with the rest included as constraints. However, in goal programming, all of the evaluative criteria can contribute to the objective function. Each objective is assigned either: (a) a preemptive priority factor which is an ordinal indication of its relative importance (Lee 1972) or (b) a cardinal weight which indicates its importance. In the latter case, goal programming essentially aggregates the multiple objectives into a single objective function.

In the former case minimization is carried out sequentially, first on the weighted deviations of the highest priority level and then down through lower priority levels. In this manner no sacrifice in goal attainment is made at the expense of a higher priority objective.

Relative weights assigned to deviational variables on the same priority level are implicitly the tradeoffs between the different objectives that the decisionmaker accepts given the optimum solution. For preemptive priority factors the decisionmaker's tradeoff ratios for objectives on different priority levels are infinite. That is, the decisionmaker would refuse any finite improvement in the attainment of a lower priority goal if it required any loss in the attainment of a higher ranked goal. Ordinal ranking of goals simply substitutes infinite relative weights for the problem of estimating finite tradeoffs. Thus, with conflicting goals assigned to multiple priority levels, one expects the attainment of lower ranked goals to be poor.

In applying goal programming to the SRS problem, preemptive priority factors were not used. Ordinal rankings are inappropriate for this problem because: (a)

preemptive goal programming does not guarantee selection of a nondominated solution (Dyer and others 1979, Field and others 1980) and (b) within the nondominated set, tradeoffs among the evaluative criteria (precision and cost) should be finite. The decisionmaker will always, where possible, be willing to reduce sample sizes slightly for a sufficiently large savings in costs or increase them slightly for sufficiently large gains in precision.

The SRS allocation problem can be formulated in a goal programming framework as shown below,

$$\begin{aligned} &\text{Minimize} && Z = w_{c1}d_c^+ + w_{c2}d_c^- + \sum_{j=1}^{v}(w_{j1}d_j^+ + w_{j2}d_j^-) \\ &\text{subject to} && V(\bar{y}_j) + d_j^- - d_j^+ = V_j \quad \text{for} \quad j = 1, 2, \ldots, v \\ & && \sum_{h=1}^{L} c_h n_h + d_c^- - d_c^+ = C - C_0 \\ & && 2 \leq n_h \leq N_h \quad \text{for} \quad h = 1, 2, \ldots, L. \end{aligned} \tag{10}$$

The goals for the variances are the V_j's and C is the cost goal. The d_c's are the deviational variables for the cost goal and the d_j's are the deviational variables for the precision goals. The w_c's and the w_j's are the corresponding weights in the objective function for these deviational variables. The last set of constraints insures that the sample sizes in each stratum allow the calculation of a stratum sample variance while not exceeding the stratum size.

Since C_0 is fixed, it can be removed from the optimization and added back to variable costs after the solution is obtained. To further simplify the problem, all of the goals (V_j's and C) can be set to zero (eliminating the negative deviational variables) as these are the "ideal" solutions (Zeleny 1976). Lastly, by replacing the n_h's with their reciprocals (labelled x_h) and filling in the equation for the $V(\bar{y}_j)$'s, the problem becomes,

$$\begin{aligned} &\text{Minimize} && Z = w_c d_c^+ + \sum_{j=1}^{v} w_{j1}d_j^+ \\ &\text{subject to} && \sum_{h=1}^{L} W_h^2 S^2_{jh} x_h - \sum_{h=1}^{L} W_h^2 S^2_{jh}/N_h - d_j^+ = 0 \\ & && \qquad\qquad \text{for} \quad j = 1, 2, \ldots, v \\ & && \sum_{h=1}^{L} c_h/x_h - d_c^+ = 0 \\ & && 1/N_h \leq x_h < \tfrac{1}{2} \quad \text{for} \quad h = 1, 2, \ldots, L. \end{aligned} \tag{11}$$

Since the problem still incorporates a nonlinear cost constraint, the most important question is whether or not it can be solved.

A general assumption of procedures for solving nonlinear programming problems is that the feasible set be convex. The requirement for convexity is that given any two feasible solutions, $X_1 = (x_{11}, x_{12}, \ldots, x_{1L})$ and $X_2 = (x_{21}, x_{22}, \ldots, x_{2L})$, all solutions of the form $\lambda_1 X_1 + \lambda_2 X_2$ (with $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 > 0$) must also belong to the feasible set. Geometrically this means that all points on a straight line between any two feasible solutions must also be feasible.

To see if this property holds for eq 11, consider the simplest possible case, simple random sampling for a single attribute ($L = v = 1$). The problem reduces to,

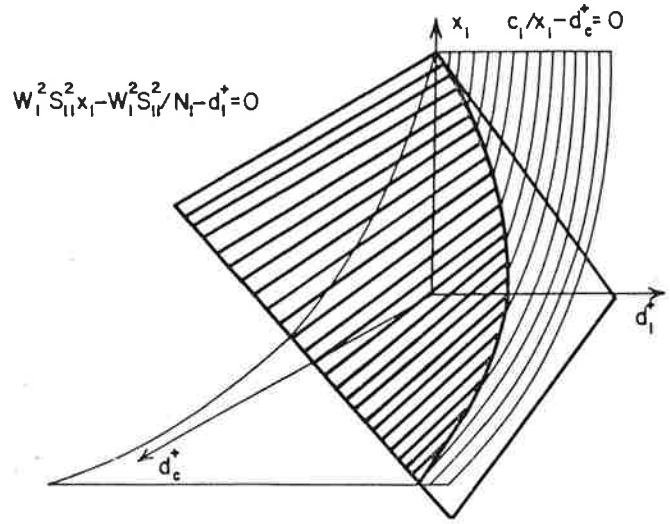

FIGURE 1. Demonstration of the nonconvex feasible set for the problem defined in eq 12.

$$\begin{aligned} &\text{Minimize} && Z = w_c d_c^+ + w_{11} d_1^+ \\ &\text{subject to} && W_1^2 S^2_{11} x_1 - W_1^2 S^2_{11}/N_1 - d_1^+ = 0 \\ & && c_1/x_1 - d_c^+ = 0 \\ & && 1/N_1 \leq x_1 \leq ½. \end{aligned} \tag{12}$$

As Figure 1 shows, the feasible region for eq 12 is a curvilinear segment which is not convex. Therefore the problem is difficult if not impossible to solve (Wagner 1975).

Fortunately, the cost constraint is separable (each decision variable occurs in a separate additive term) so that b-1 linear segments with b break points can be used to approximate it as shown in Figure 2. In each stratum, x_h is replaced by $\sum_{m=1}^{b} \lambda_{hm}/X_{hm}$ with $\sum_{m=1}^{b} \lambda_{hm} = 1$ and all λ's nonnegative. The λ's are unknown weights for the linear combinations of the X's. These X's are constants such that $X_{hm} = 1/x_h$ evaluated at the mth breakpoint for the linear approximation to the cost function in the hth stratum. In order to restrict the solution to one of the linear segments shown in Figure 2, no more than two of the λ's for each stratum may be greater than zero, and if two are, they must be adjacent. Temporarily ignoring this last restriction, the approximation produces the following problem,

$$\begin{aligned} &\text{Minimize} && Z = w_c d_c^+ + \sum_{j=1}^{v} w_{j1} d_j^+ \\ &\text{subject to} && \sum_{h=1}^{L} W_h^2 S^2_{jh} \left(\sum_{m=1}^{b} \lambda_{hm}/X_{hm} \right) - \sum_{h=1}^{L} W_h^2 S^2_{jh}/N_h - d_j^+ = 0 \\ & && \text{for } j = 1, 2, \ldots, v \\ & && \sum_{h=1}^{L} c_h \left(\sum_{m=1}^{b} \lambda_{hm} X_{hm} \right) - d_c^- = 0 \end{aligned} \tag{13}$$

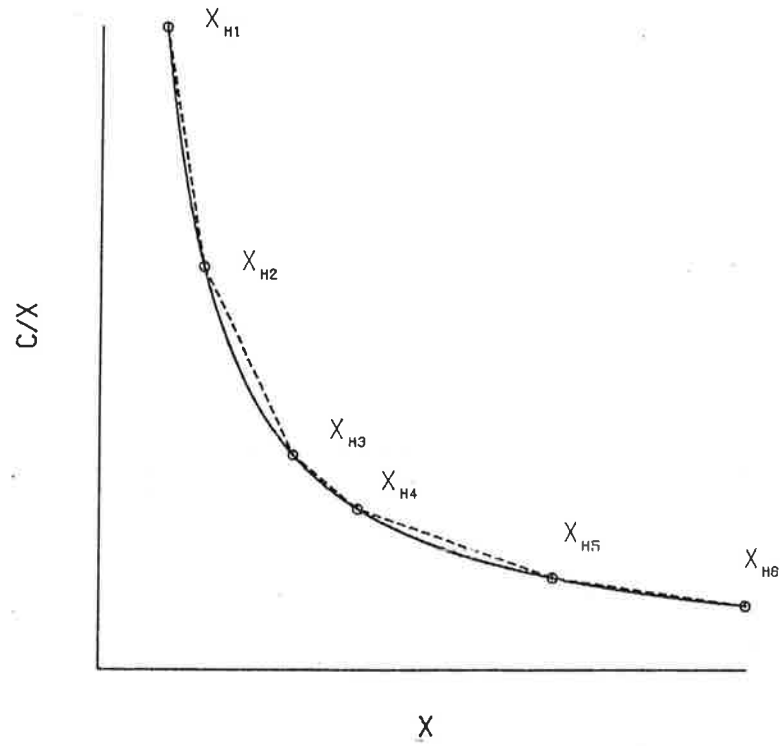

FIGURE 2. Approximation of each term of the cost functional by linear (dashed) segments.

$$1/N_h \leq \left(\sum_{m=1}^{b} \lambda_{hm}/X_{hm} \right) \leq 1/2$$

$$\sum_{m=1}^{b} \lambda_{hm} = 1 \quad \text{for} \quad h = 1, 2, \ldots, L.$$

Note that the x_h's in the precision and sample size constraints are all replaced with the appropriate linear approximations as well. This approximation is linear in the λ's and d's and can be solved by linear programming if the restriction about adjacent weights (λ's) is ignored. The property of adjacent weights insures that this restriction will, in fact, be forced upon the solution by the solution procedure (Wagner 1975). Because of this property, the feasible set will have a linear cost constraint and thus will be convex.

As the number of linear segments increases (at least in the neighborhood of the optimal solution), the optimum for the approximated problem (eq 13) approaches the true optimum for the original problem (eq 11) (Simmons 1975). Increasing the number of segments enlarges the number of decision variables and increases the number of iterations needed to find the solution. One approach is to iteratively solve the problem forming a finer grid of segments until a stable optimum is found. Another approach is to retain the same number of segments for each iteration but to shift the segments to form a finer approximation in the neighborhood of the optimum. For the application discussed in the next section, the latter method was adopted.

TABLE 1. Summary of stratum codes and stand structure.

Stratum code	Stand structure
Commercial forest land	
21	Softwoods, not ready for harvest, sparsely stocked
22	Softwoods, not ready for harvest, adequately stocked
23	Softwoods, not ready for harvest, overstocked
24	Softwoods, commercial size, adequate regeneration
25	Softwoods, commercial size, understory overstocked
26	Softwoods, commercial size, understory not adequately stocked
27	Softwoods, seedlings or saplings
31	Cottonwood and aspen
Noncommercial forest land	
41	Softwoods, poor site, some volume
42	Softwoods, poor site, little volume
43	Hardwoods, poor site, little volume
44	Pinyon and juniper
60	Nonforest, land
92	Nonforest, water

Application of the Goal Programming Method to Forest Inventory

The goal programming formulation was tested on data from a recent U.S. Forest Service Inventory of three counties in northern New Mexico.[2] The basic sampling design was a stratified double sample using stratum weights estimated from aerial photographs.

The primary sample consisted of one-acre photo points from a systematic grid with a 1 percent sampling intensity. Each photo point was classified by stand type into one of the fourteen strata defined in Table 1 and the proportion of photo points belonging to each stratum was used as an estimate of the stratum weight, W_h.[3]

The secondary sample consisted of ground points taken at a subset of the photo points. At each ground location a ten-point cluster of prism points was established. The average cost for each ground location was $130. The ground observations indicated that stand type classifications based on aerial photography were in error for many points.[4]

The Forest Service technique for determining sample sizes assumes proportional sampling across strata with the total sample size (n) fixed by an allowable standard error of the estimate. The proportional allocations of sampling effort were computed with respect to estimates of: (a) growing stock volume, (b) area of commercial forest land (CFL), and (c) area of noncommercial forest land (NCFL). The largest of the three sample sizes for a stratum was selected as the final allocation for that stratum. This procedure is a reasonable compromise so-

[2] Ibid.

[3] Since estimates of stratum weights were used, the expression for the $V(\bar{y})$ used in eq 13 was replaced by the following equation due to Bickford and others (1963),

$$V(\bar{y}) = [N(N-1)]^{-1}\left[\sum_{h=1}^{L}(N_h - 1)N_h S^2_{jh}/n_h + N_h(\bar{y}_{jh} - \bar{y}_j)^2\right]. \quad (14)$$

[4] Mitchell, B. 1978. Optimization of multivariate stratified random sampling designs for forest inventory. Unpublished MS thesis, Univ Washington, Seattle, Wash. 64 p.

lution but it ignores other characteristics of interest and differences in sampling costs and variation among strata.

To illustrate the use of the goal programming method, the following characteristics were arbitrarily chosen for estimation from the dozens of variables measured at each ground point: (a) average potential yield, (b) growth, (c) mortality, (d) cull volume, and (e) growing stock volume. Characteristics (a)–(c) were measured in CF/A/Yr whereas characteristics (d) and (e) were expressed in CF/A.

For each characteristic the sample means and variances were estimated using the data from the Forest Service inventory. For lack of better information on costs, it was arbitrarily assumed that each ground location cost $200 in CFL and $50 in NCFL. (Nonforest land points were not visited on the ground.)

The approximation to eq 13 (using 10 breakpoints) for 14 strata was

$$\text{Minimize} \qquad Z = w_c d_c^+ + \sum_{j=1}^{5} w_{j1} d_j^+$$

subject to

$$\left\{[N(N-1)]^{-1} \sum_{h=1}^{14} (N_h - 1) N_h S^2_{jh} \left(\sum_{m=1}^{10} \lambda_{hm}/X_{hm} \right) + N_h(\bar{y}_{jh} - \bar{y}_j)^2 \right\} - d_j^+ = 0$$

$$\text{for} \quad j = 1, 2, \ldots, 5 \tag{15}$$

$$\sum_{h=1}^{14} c_h \left(\sum_{m=1}^{10} \lambda_{hm} X_{hm} \right) - d_c^+ = 0$$

$$1/N_h \leq \sum_{m=1}^{10} \lambda_{hm}/X_{hm} \leq ½$$

$$\sum_{m=1}^{10} \lambda_{hm} = 1 \quad \text{for} \quad h = 1, 2, \ldots, 14.$$

This problem was then re-solved with the linear segments shifted for a finer approximation in the neighborhood of the optimum, until the relative difference between the approximated costs and the true costs of the generated solution were less than 5 percent. The use of nine segments (ten breakpoints such as shown in Fig. 2) was found to yield a good compromise between problem size and the number of iterations required for a good approximation to the solution.

As an example, results for one of the three counties—Sante Fe—are presented. Table 2 lists the deviational weights and the resulting coefficients of variation (CV) of each characteristic for eight possible solutions to the SRS problem (eq 15). The corresponding sampling allocations determined by these sets of weights are displayed in Table 3. Although eq 15 is stated in terms of $V(\bar{y}_j)$, the information on precision of the estimates is presented in terms of CV to aid the decisionmaker in comparing the achievement of precision goals within any one solution. Unlike the variance, the CV is scale free. Since stratum 92 was less than two sampling units in size ($N_h = 1.92$), the lower constraint on n_h for this stratum was reduced from 2 to 1 to keep the problem feasible.

The starting point for generating sampling allocations was to set each element of the vector of deviational weights equal to one. This set of weights resulted in Plan A, which took the minimal sample size in each stratum yielding the minimum cost plan. The opposite extreme was Plan B which took maximum sample sizes in each stratum that had nonzero tree volume (strata 21–27 and 41). Plan B was the maximum precision solution as the maximum sample sizes minimized the CV's for all characteristics. Even with complete enumeration, the CV's (and thus the variances) for each characteristic do not reduce to zero for Plan B because the variances as given in eq 14 contain a positive term due to the estimation of the W_h's.

TABLE 2. Summary of sampling plans for Santa Fe County (plans A–H).

Precision of estimate of—	Plan A		Plan B		Plan C		Plan D	
	GW[1]	CV[2]	GW	CV	GW	CV	GW	CV
Average yield	1.00	46.52	1.00	9.06	1.00	17.59	1.00	12.48
Growth/year	1.00	44.66	1.00	13.17	1.00	26.01	1.00	28.03
Mortality/year	1.00	300.83	1.00	36.60	1.00	99.68	10.00	46.17
Cull volume	1.00	101.70	1.00	14.39	1.00	35.90	1.00	23.62
Growing stock volume	1.00	50.47	1.00	11.63	1.00	23.04	1.00	21.37
Cost	1.00	$3,600	0.000001	$24,017	0.001	$4,926	0.0001	$11,772

Precision of estimate of—	Plan E		Plan F		Plan G		Plan H	
	GW	CV	GW	CV	GW	CV	GW	CV
Average yield	1.00	17.53	1.00	17.47	0.00	17.38	—	14.42
Growth/year	1.00	25.73	1.00	25.36	0.00	24.80	—	15.89
Mortality/year	10.00	99.68	10.00	99.68	0.00	99.68	—	83.14
Cull volume	2.00	35.83	2.00	35.65	0.00	35.44	—	28.68
Growing stock volume	2.00	22.88	2.00	22.67	1.00	22.53	—	16.38
Cost	0.0018	$4,971	0.0017	$5,036	0.0007	$5,147	—	$14,800

[1] Goal weight.
[2] Coefficient of variation.

TABLE 3. Summary of sample allocations for Santa Fe County (plans A–H).

Stratum number	Stratum size in photo points	Ground sample size							
		Plan A	Plan B	Plan C	Plan D	Plan E	Plan F	Plan G	Plan H
Commercial forest land									
21	6.44	2.00	6.44	2.00	2.00	2.00	2.00	2.00	5.00
22	25.72	2.00	25.72	3.65	2.04	3.65	3.98	4.31	25.00
23	5.84	2.00	5.84	2.00	2.00	2.00	2.00	2.00	5.00
24	10.40	2.00	10.40	2.00	2.00	2.00	2.00	2.00	9.00
25	5.13	2.00	5.13	2.00	2.00	2.00	2.00	2.00	5.00
26	18.17	2.00	18.17	2.45	2.03	2.67	2.67	2.90	18.00
27	3.60	2.00	3.60	2.00	2.00	2.00	2.00	2.00	3.00
31	2.52	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
Noncommercial forest land									
41	165.16	2.00	165.16	20.13	165.16	20.13	20.13	20.13	27.00
42	21.94	2.00	2.00	2.00	2.00	2.00	2.00	2.00	10.00
43	29.41	2.00	2.00	2.00	2.00	2.00	2.00	2.00	10.00
44	2,580.38	2.00	2.00	2.00	2.00	2.00	2.00	2.00	10.00
Nonforest, land									
60	4,695.26	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
Nonforest, water									
92	1.26	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Plan C is a compromise between the cost extremes of A and B, still keeping all deviational weights for precision goals equal to one.

Plans D, E, and F illustrate how changes in the weights for the precision objectives shift the sampling allocations among the strata to adjust CV's. In these plans, the weights for mortality, cull volume, and growing stock volume were increased to find solutions with lower CV's for these characteristics. The resulting solutions, however, reduced the CV's of all the characteristics, reflecting the correlations within strata among these characteristics. Plan G was very similar in cost and precision to plans E and F but was generated by a radically different set of weights. This points out how correlations among characteristics can limit the effects of major shifts in deviational weights on the resulting sampling allocation. It also suggests that if there are groups of strongly correlated characteristics to be estimated by an SRS design, one representative characteristic should be selected from each group and used when exploring the solution space. The CV's of the other characteristics in a group can be assumed to be approximately proportional to that of the representative variable for a given allocation.

The actual Forest Service plan (Plan H) allocates considerably more sampling effort to strata with low and zero tree volumes in order to meet constraints on the allowable error for area estimates. Since these goals differ from those incorporated in the case study, Plan H is not comparable to Plans A–G.

A goal programming approach can be adapted to search for a best compromise solution for many sampling designs, only requiring that any nonlinear cost or variance functions be separable in the decision variables. For example, the combined problem of choosing the number of photo points in the primary sample as well as in the secondary sample would fit into this framework. Two-stage sampling designs would also be amenable though the number of nonlinearities in the problem increases. For very complex designs, the size of the linearized problem may be so large that the costs of exploring the solution space in this manner could be prohibitively expensive.

The simplest extension of this method would be to modify the linear cost function by adding terms of the form $c_h/x_h^{1/2}$ as suggested by Sukhatme (1954). This would take into account travel costs between ground sample points. Unfortunately, cost functions are very difficult to estimate adequately as standardization of methodology and operating conditions are lacking.

Discussion

The separable goal programming approach offers a number of advantages over previous techniques for solving the multivariate SRS allocation problem. Although it utilizes highly efficient LP solution algorithms, it always produces a nondominated solution since the goal attainment levels are set as high as possible (ideally). Conceptually it is appealing because all decision criteria are used in an identical manner, without the somewhat artificial division into objective function and constraints. Lastly, the approach does not presuppose that the decisionmaker's preference structure is known before any solutions are presented. Instead, it allows one to search the solution space in a directed manner towards an acceptable best compromise solution.

The shortcomings of the approach, shared by all optimization techniques for the problem, include assumptions of perfect knowledge of stratum means and variances. In practice, these quantities must be estimated, either from a previous inventory if the same classification of strata is being used, or possibly from aerial photos of the primary sample. Further, it is difficult to estimate the cost functions needed to implement the solution procedure. Consequently, the utility of the solutions to the SRS problem generated by the goal programming approach will be limited by the accuracy of these estimates.

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